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Linear mappings derivable at some nontrivial elements ☆

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ABSTRACT

Suppose that $\mathcal{A}$ is an algebra and $\mathcal{M}$ is an $\mathcal{A}$-bimodule. Let A be any element in $\mathcal{A}$. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is said to be derivable at A if $\delta(ST) = \delta(S)T + S\delta(T)$ for any S, T in $\mathcal{A}$ with $ST = A$. Given an algebra $\mathcal{A}$, such as a non-abelian von Neumann algebra or an irreducible CDCSL algebra on a Hilbert space H with *dimH* $\geqslant 2$, we show that there exists a nontrivial idempotent P in $\mathcal{A}$ such that for any $Q \in P\mathcal{A}P$ which is invertible in $P\mathcal{A}P$, every linear mapping derivable at Q from $\mathcal{A}$ into some unital $\mathcal{A}$-bimodule (for example, $\mathcal{A}$ or $B(H)$) is derivation.

1. Introduction

Let $\mathcal{A}$ be an algebra over a (real or complex) field and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for any $A, B \in \mathcal{A}$. Let $A \in \mathcal{A}$, a linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is said to be *derivable at* A if $\delta(ST) = \delta(S)T + S\delta(T)$ for any $S, T \in \mathcal{A}$ with $ST = A$. We say an element A in $\mathcal{A}$ is an *all-derivable point of* $\mathcal{A}$ if every linear mapping derivable at A from $\mathcal{A}$ into $\mathcal{A}$ is a derivation.

In recent years, several authors have studied the mappings derivable at some points. Papers [2,4,9] study the linear mappings derivable at zero. Papers [8,13,21,22] study the linear mappings derivable at some invertible elements. In [23], Zhu and Xiong show that for some nest $\mathcal{N}$ and $0 \neq M \in \mathcal{N}$, the orthogonal projection onto M is an all-derivable point of alg$\mathcal{N}$ for the strong operator topology. In [3], authors show that under some conditions on triangular ring $\mathcal{I}$, some idempotents of $\mathcal{I}$ are additive all-derivable points. In [16], under certain conditions, Lu shows that every linear mapping derivable at some idempotents from any unital Banach algebra into its unital bimodule is a derivation.

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In this paper, we want to discuss the following problem.

Question. Given a unital algebra $\mathcal{A}$ with a nontrivial idempotent P and a unital $\mathcal{A}$-bimodule $\mathcal{M}$. Let $Q \in P\mathcal{A}P$ be invertible in $P\mathcal{A}P$, is it true that every linear mapping derivable at Q from $\mathcal{A}$ into $\mathcal{M}$ is a derivation?

To deal with the question as much as possible, in the following, we will introduce another construction to define the unital algebras with at least one nontrivial idempotent. Firstly, we give some concepts.

We call $\mathcal{A}$ a *unital algebra* if $\mathcal{A}$ is an algebra with identity. We use the symbols $I_{\mathcal{A}}$ to denote the unit element in $\mathcal{A}$. $\mathcal{M}$ is called a *unital $\mathcal{A}$-bimodule* if $\mathcal{M}$ is an $\mathcal{A}$-bimodule with $I_{\mathcal{A}}m = mI_{\mathcal{A}} = m$ for all m in $\mathcal{M}$.

Let $\mathcal{U}$ be any unital algebra with at least one nontrivial idempotent P and $\mathcal{V}$ be any unital $\mathcal{U}$-bimodule. We can view $\mathcal{U}$ as $\begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{W} & \mathcal{B} \end{pmatrix}$, where $\mathcal{A} = P\mathcal{U}P$, $\mathcal{M} = P\mathcal{U}(I_{\mathcal{U}} - P)$, $\mathcal{W} = (I_{\mathcal{U}} - P)\mathcal{U}P$, $\mathcal{B} = (I_{\mathcal{U}} - P)\mathcal{U}(I_{\mathcal{U}} - P)$ and $\mathcal{V}$ as $\begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_3 & \mathcal{M}_4 \end{pmatrix}$, where $\mathcal{M}_1 = P\mathcal{V}P$, $\mathcal{M}_2 = P\mathcal{V}(I_{\mathcal{U}} - P)$, $\mathcal{M}_3 = (I_{\mathcal{U}} - P)\mathcal{V}P$, $\mathcal{M}_4 = (I_{\mathcal{U}} - P)\mathcal{U}(I_{\mathcal{U}} - P)$. In this paper, we still identify $\begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{W} & \mathcal{B} \end{pmatrix}$ by $\mathcal{U}$ and $\begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_3 & \mathcal{M}_4 \end{pmatrix}$ by $\mathcal{V}$ and we call them *unital block algebra* and *unital block $\mathcal{U}$-bimodule* respectively. Clearly, $P = \begin{pmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}$ and we call it *the standard idempotent of $\mathcal{U}$*.

In Section 2, under some conditions, we show that every linear mapping derivable at some nontrivial element from a unital block algebra $\mathcal{U}$ into unital block $\mathcal{U}$-bimodule $\mathcal{V}$ is a derivation and we generalize the main results in [3,16].

In Sections 3 and 4, we use the results in Section 2 to many kinds of unital operator algebras and we show that for such kinds of unital operator algebras, the answer of above question is "Yes". Of course, we also give several cases such that the answer of above question is "No". In particular, we generalize the main result in [20] without any assumption of continuity.

2. Linear mapping derivables at the standard idempotent on unital block algebras

Lemma 2.1. *Suppose that $\mathcal{U} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{W} & \mathcal{B} \end{pmatrix}$ is a unital block algebra which satisfies that for any A in $\mathcal{A}$, there is some integer n such that $nI_{\mathcal{A}} - A$ is invertible in $\mathcal{A}$, $\mathcal{V} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_3 & \mathcal{M}_4 \end{pmatrix}$ is a unital block $\mathcal{U}$-bimodule. Let $Q = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$, where $J \in \mathcal{A}$ and J is invertible in $\mathcal{A}$. If δ is a linear mapping derivable at Q from $\mathcal{U}$ into $\mathcal{V}$, then there are elements $D \in \mathcal{M}_2$, $C \in \mathcal{M}_3$ and linear mappings $a_{11} : \mathcal{A} \to \mathcal{M}_1$, $b_{12} : \mathcal{M} \to \mathcal{M}_2$, $c_{21} : \mathcal{W} \to \mathcal{M}_3$, $d_{22} : \mathcal{B} \to \mathcal{M}_4$ such that a_{11} is a linear mapping derivable at J, d_{22} is a linear mapping derivable at zero and*

$$\delta\left(\begin{pmatrix} A & M \\ W & B \end{pmatrix}\right) = \begin{pmatrix} a_{11}(A) - MC - DW & AD + b_{12}(M) - DB \\ CA + c_{21}(W) - BC & CM + WD + d_{22}(B) \end{pmatrix}, \tag{2.1}$$

$$b_{12}(MB) = b_{12}(M)B + Md_{22}(B), \tag{2.2}$$

$$b_{12}(AM) = a_{11}(A)M + Ab_{12}(M), \tag{2.3}$$

$$c_{21}(BW) = d_{22}(B)W + Bc_{21}(W), \tag{2.4}$$
$$c_{21}(WA) = c_{21}(W)A + Wa_{11}(A), \tag{2.5}$$
$$a_{11}(MW) = b_{12}(M)W + Mc_{21}(W), \tag{2.6}$$
$$(d_{22}(WM) - c_{21}(W)M - Wb_{12}(M))W' = 0, \tag{2.7}$$
$$M'(d_{22}(WM) - c_{21}(W)M - Wb_{12}(M)) = 0 \tag{2.8}$$

for any $A \in \mathcal{A},\ M, M' \in \mathcal{M},\ W, W' \in \mathcal{W}$ *and* $B \in \mathcal{B}$.

Proof. For any $A \in \mathcal{A}$, $M \in \mathcal{M}$, $W \in \mathcal{W}$, $B \in \mathcal{B}$, we write

$$\delta\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} a_{11}(A) & a_{12}(A) \\ a_{21}(A) & a_{22}(A) \end{pmatrix},$$
$$\delta\left(\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} b_{11}(M) & b_{12}(M) \\ b_{21}(M) & b_{22}(M) \end{pmatrix},$$
$$\delta\left(\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}\right) = \begin{pmatrix} c_{11}(W) & c_{12}(W) \\ c_{21}(W) & c_{22}(W) \end{pmatrix},$$
$$\delta\left(\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}\right) = \begin{pmatrix} d_{11}(B) & d_{12}(B) \\ d_{21}(B) & d_{22}(B) \end{pmatrix},$$

respectively.

Step 1. Let $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$ satisfy $A_1A_2 = J$ and $B_1B_2 = 0$. If we take $S = \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}$, $T = \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}$, then $ST = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} = Q$. Since δ is a linear mapping derivable at Q from $\mathcal{U}$ into $\mathcal{V}$, we have that

$$\begin{pmatrix} a_{11}(J) & a_{12}(J) \\ a_{21}(J) & a_{22}(J) \end{pmatrix} = \delta(Q) = \delta(S)T + S\delta(T)$$
$$= \begin{pmatrix} a_{11}(A_1) + d_{11}(B_1) & a_{12}(A_1) + d_{12}(B_1) \\ a_{21}(A_1) + d_{21}(B_1) & a_{22}(A_1) + d_{22}(B_1) \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}$$
$$+ \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} a_{11}(A_2) + d_{11}(B_2) & a_{12}(A_2) + d_{12}(B_2) \\ a_{21}(A_2) + d_{21}(B_2) & a_{22}(A_2) + d_{22}(B_2) \end{pmatrix}$$
$$= \begin{pmatrix} \begin{matrix} a_{11}(A_1)A_2 + d_{11}(B_1)A_2 \\ +A_1a_{11}(A_2) + A_1d_{11}(B_2) \end{matrix} & \begin{matrix} a_{12}(A_1)B_2 + d_{12}(B_1)B_2 \\ +A_1a_{12}(A_2) + A_1d_{12}(B_2) \end{matrix} \\ \\ \begin{matrix} a_{21}(A_1)A_2 + d_{21}(B_1)A_2 \\ +B_1a_{21}(A_2) + B_1d_{21}(B_2) \end{matrix} & \begin{matrix} a_{22}(A_1)B_2 + d_{22}(B_1)B_2 \\ +B_1a_{22}(A_2) + B_1d_{22}(B_2) \end{matrix} \end{pmatrix}. \tag{2.9}$$

If we take $B_1 = B_2 = 0$ in (2.9), then

$$a_{11}(J) = a_{11}(A_1)A_2 + A_1a_{11}(A_2), \tag{2.10}$$
$$a_{12}(J) = A_1a_{12}(A_2), \tag{2.11}$$
$$a_{21}(J) = a_{21}(A_1)A_2, \tag{2.12}$$
$$a_{22}(J) = 0 \tag{2.13}$$

for any $A_1, A_2 \in \mathcal{A}$ with $A_1A_2 = J$. For any $A \in \mathcal{A}$, there is an integer n such that $F = nI_{\mathcal{A}} - A$ is invertible in $\mathcal{A}$. By (2.11) and (2.12), we have that

$$Ja_{12}(I_{\mathcal{A}}) = a_{12}(J) = JF^{-1}a_{12}(F),$$
$$a_{21}(I_{\mathcal{A}})J = a_{21}(J) = a_{21}(F)F^{-1}J.$$

Since J is invertible in $\mathcal{A}$ and $F = nI_{\mathcal{A}} - A$, we have that

$$a_{12}(A) = Aa_{12}(I_{\mathcal{A}}), \tag{2.14}$$
$$a_{21}(A) = a_{21}(I_{\mathcal{A}})A \tag{2.15}$$

for any $A \in \mathcal{A}$. **We abbreviate** $D = a_{12}(I_{\mathcal{A}})$ and $C = a_{21}(I_{\mathcal{A}})$.

If we take $A_1 = J$, $A_2 = I_{\mathcal{A}}$, $B_1 = B$ and $B_2 = 0$ in (2.9), then

$$d_{11}(B) = -Ja_{11}(I_{\mathcal{A}}), \tag{2.16}$$
$$d_{21}(B) = -Ba_{21}(I_{\mathcal{A}}) = -BC, \tag{2.17}$$
$$Ba_{22}(I_{\mathcal{A}}) = 0 \tag{2.18}$$

for any $B \in \mathcal{B}$. If we take $B = 0$ in (2.16), then $Ja_{11}(I_{\mathcal{A}}) = 0$. Thus

$$a_{11}(I_{\mathcal{A}}) = 0. \tag{2.19}$$

By (2.16) and (2.19), we have that

$$d_{11}(B) = 0 \tag{2.20}$$

for any $B \in \mathcal{B}$. If we take $B = I_{\mathcal{B}}$ in (2.18), then

$$a_{22}(I_{\mathcal{A}}) = 0. \tag{2.21}$$

If we take $A_1 = J$, $A_2 = I_{\mathcal{A}}$, $B_1 = 0$ and $B_2 = B$ in (2.9), then

$$Jd_{12}(B) = -a_{12}(J)B = -JDB$$

for any $B \in \mathcal{B}$. Thus

$$d_{12}(B) = -DB \tag{2.22}$$

for any $B \in \mathcal{B}$.

If we take $B_1 = 0$ and $B_2 = I_{\mathcal{B}}$ in (2.9), then by (2.13),

$$a_{22}(F) = a_{22}(J) = 0 \tag{2.23}$$

for any invertible element F in $\mathcal{A}$. For any $A \in \mathcal{A}$, there is an integer n such that $nI_{\mathcal{A}} - A$ is invertible in $\mathcal{A}$. By (2.21) and (2.23), we have that

$$a_{22}(A) = 0, \tag{2.24}$$

for any $A \in \mathcal{A}$.

If we take $A_1 = J$ and $A_2 = I_{\mathcal{A}}$ in (2.9), then by (2.24),

$$d_{22}(B_1)B_2 + B_1d_{22}(B_2) = 0 \tag{2.25}$$

for any $B_1, B_2 \in \mathcal{B}$ with $B_1B_2 = 0$.

Step 2. For any $A \in \mathcal{A}$, there is an integer n such that $nI_{\mathcal{A}} - A$ is invertible in $\mathcal{A}$. Denote $F = nI_{\mathcal{A}} - A$. For any $M \in \mathcal{M}$ and $B \in \mathcal{B}$, if we take $S = \begin{pmatrix} F & -FM \\ 0 & 0 \end{pmatrix}$, $T = \begin{pmatrix} F^{-1}J & MB \\ 0 & B \end{pmatrix}$, then $ST = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} = Q$. It follows from (2.10), (2.14)–(2.17) and(2.20)–(2.24) that

$$\begin{aligned}
&\begin{pmatrix} a_{11}(J) & JD \\ CJ & 0 \end{pmatrix} = \delta(Q) = \delta(S)T + S\delta(T) \\
&= \begin{pmatrix} a_{11}(F) - b_{11}(FM) & FD - b_{12}(FM) \\ CF - b_{21}(FM) & -b_{22}(FM) \end{pmatrix} \begin{pmatrix} F^{-1}J & MB \\ 0 & B \end{pmatrix} \\
&\quad + \begin{pmatrix} F & -FM \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11}(F^{-1}J) + b_{11}(MB) & F^{-1}JD + b_{12}(MB) - DB \\ CF^{-1}J + b_{21}(MB) - BC & b_{22}(MB) + d_{22}(B) \end{pmatrix} \\
&= \begin{pmatrix} \begin{array}{c} a_{11}(J) - b_{11}(FM)F^{-1}J \\ +Fb_{11}(MB) - FMCF^{-1}J \\ -FMb_{21}(MB) + FMBC \end{array} & \begin{array}{c} a_{11}(F)MB - b_{11}(FM)MB \\ -b_{12}(FM)B + JD + Fb_{12}(MB) \\ -FMb_{22}(MB) - FMd_{22}(B) \end{array} \\ \\ CJ - b_{21}(FM)F^{-1}J & \begin{array}{c} CFMB - b_{21}(FM)MB \\ -b_{22}(FM)B \end{array} \end{pmatrix}. \qquad (2.26)
\end{aligned}$$

If we take $F = I_{\mathcal{A}}$ and $B = 0$ in (2.26), then

$$b_{21}(M)J = 0,$$
$$-b_{11}(M)J - MCJ = 0$$

for any $M \in \mathcal{M}$. Thus

$$b_{21}(M) = 0, \qquad (2.27)$$
$$b_{11}(M) = -Ma_{21}(I_{\mathcal{A}}) = -MC \qquad (2.28)$$

for any $M \in \mathcal{M}$.

If we take $F = I_{\mathcal{A}}$ in (2.26), then

$$CMB - b_{22}(M)B = 0, \qquad (2.29)$$
$$-b_{11}(M)MB - b_{12}(M)B + b_{12}(MB) - Mb_{22}(MB) - Md_{22}(B) = 0. \qquad (2.30)$$

If we take $B = I_{\mathcal{B}}$ in (2.29), then we have that

$$b_{22}(M) = CM \qquad (2.31)$$

for any $M \in \mathcal{M}$. By (2.28), (2.30) and (2.31), we have that

$$\begin{aligned} 0 &= MCMB - b_{12}(M)B + b_{12}(MB) - MCMB - Md_{22}(B) \\ &= b_{12}(MB) - b_{12}(M)B - Md_{22}(B) \end{aligned} \qquad (2.32)$$

for any $M \in \mathcal{M}$, $B \in \mathcal{B}$. If we take $B = I_{\mathcal{B}}$ in (2.32), then

$$Md_{22}(I_{\mathcal{B}}) = 0 \qquad (2.33)$$

for any $M \in \mathcal{M}$.

If we take $B = I_{\mathcal{B}}$ in (2.26), then

$$a_{11}(F)M - b_{11}(FM)M - b_{12}(FM) + Fb_{12}(M) - FMb_{22}(M) - FMd_{22}(I_{\mathcal{B}}) = 0 \tag{2.34}$$

for any $M \in \mathcal{M}$. It follows from (2.28), (2.31), (2.33) and (2.34) that

$$\begin{aligned} 0 &= a_{11}(F)M + FMCM - b_{12}(FM) + Fb_{12}(M) - FMCM \\ &= a_{11}(F)M - b_{12}(FM) + Fb_{12}(M). \end{aligned}$$

Since $a_{11}(I_{\mathcal{A}}) = 0$, we have that

$$a_{11}(A)M - b_{12}(AM) + Ab_{12}(M) = 0 \tag{2.35}$$

for any $A \in \mathcal{A}$, $M \in \mathcal{M}$.

Step 3. For any $A \in \mathcal{A}$, there is an integer n such that $nI_{\mathcal{A}} - A$ is invertible in $\mathcal{A}$. Denote $F = nI_{\mathcal{A}} - A$. For any $W \in \mathcal{W}$, $B \in \mathcal{B}$, if we take $S = \begin{pmatrix} JF^{-1} & 0 \\ BW & B \end{pmatrix}$, $T = \begin{pmatrix} F & 0 \\ -WF & 0 \end{pmatrix}$, then $ST = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} = Q$. It follows from (2.10), (2.14)–(2.17) and (2.20)–(2.24) that

$$\begin{aligned} \begin{pmatrix} a_{11}(J) & JD \\ CJ & 0 \end{pmatrix} &= \delta(Q) = \delta(S)T + S\delta(T) \\ &= \begin{pmatrix} a_{11}(JF^{-1}) + c_{11}(BW) & JF^{-1}D + c_{12}(BW) - DB \\ CJF^{-1} + c_{21}(BW) - BC & c_{22}(BW) + d_{22}(B) \end{pmatrix} \begin{pmatrix} F & 0 \\ -WF & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} JF^{-1} & 0 \\ BW & B \end{pmatrix} \begin{pmatrix} a_{11}(F) - c_{11}(WF) & FD - c_{12}(WF) \\ CF - c_{21}(WF) & -c_{22}(WF) \end{pmatrix} \\ &= \begin{pmatrix} \begin{array}{l} a_{11}(J) + c_{11}(BW)F \\ -JF^{-1}DWF - c_{12}(BW)WF \\ +DBWF - JF^{-1}c_{11}(WF) \end{array} & JD - JF^{-1}c_{12}(WF) \\ \begin{array}{l} CJ + c_{21}(BW)F - c_{22}(BW)WF \\ -d_{22}(B)WF + BWa_{11}(F) \\ -BWc_{11}(WF) - Bc_{21}(WF) \end{array} & \begin{array}{c} BWFD - BWc_{12}(WF) \\ -Bc_{22}(WF) \end{array} \end{pmatrix}. \end{aligned} \tag{2.36}$$

If we take $F = I_{\mathcal{A}}$ and $B = 0$ in (2.36), then

$$\begin{aligned} &Jc_{12}(W) = 0, \\ &Jc_{11}(W) = -Ja_{12}(I_{\mathcal{A}})W = -JDW \end{aligned}$$

for any $W \in \mathcal{W}$. Thus

$$c_{12}(W) = 0, \tag{2.37}$$

$$c_{11}(W) = -a_{12}(I_{\mathcal{A}})W = -DW \tag{2.38}$$

for any $W \in \mathcal{W}$.

If we take $F = I_{\mathcal{A}}$ in (2.36), then

$$BWD - Bc_{22}(W) = 0 \tag{2.39}$$

$$c_{21}(BW) - c_{22}(BW)W - d_{22}(B)W - BWc_{11}(W) - Bc_{21}(W) = 0 \tag{2.40}$$

for any $W \in \mathcal{W}$ and $B \in \mathcal{B}$. If we take $B = I_{\mathcal{B}}$ in (2.39), then

$$c_{22}(W) = WD \tag{2.41}$$

for any $W \in \mathcal{W}$. It follows from (2.38), (2.40) and (2.41) that

$$\begin{aligned} 0 &= c_{21}(BW) - BWDW - d_{22}(B)W + BWDW - Bc_{21}(W) \\ &= c_{21}(BW) - d_{22}(B)W - Bc_{21}(W) \end{aligned} \tag{2.42}$$

for any $W \in \mathcal{W}$ and $B \in \mathcal{B}$.

If we take $B = I_{\mathcal{B}}$ in (2.36), then

$$c_{21}(W)F - c_{22}(W)WF - d_{22}(I_{\mathcal{B}})WF + Wa_{11}(F) + Wc_{11}(WF) - c_{21}(WF) = 0. \tag{2.43}$$

It follows from (2.38), (2.39) and (2.43)

$$\begin{aligned} 0 &= c_{21}(W)F - WDWF - d_{22}(I_{\mathcal{B}})WF + Wa_{11}(F) + WDWF - c_{21}(WF) \\ &= c_{21}(W)F + Wa_{11}(F) - c_{21}(WF) - d_{22}(I_{\mathcal{B}})WF. \end{aligned} \tag{2.44}$$

If we take $F = I_{\mathcal{A}}$ in (2.44), then it follows from (2.14) that

$$d_{22}(I_{\mathcal{B}})W = Wa_{11}(I_{\mathcal{A}}) = 0 \tag{2.45}$$

for any $W \in \mathcal{W}$. Thus, by (2.44) and (2.45), $c_{21}(W)F + Wa_{11}(F) - c_{21}(WF) = 0$. Since $a_{11}(I_{\mathcal{A}}) = 0$, we have that

$$c_{21}(W)A + Wa_{11}(A) - c_{21}(WA) = 0 \tag{2.46}$$

for any $A \in \mathcal{A}$, $W \in \mathcal{W}$.

Step 4. For any $M \in \mathcal{M}, W \in \mathcal{W}$, if we take $S = \begin{pmatrix} I_{\mathcal{A}} - MW & M \\ 0 & 0 \end{pmatrix}$, $T = \begin{pmatrix} J & 0 \\ WJ & 0 \end{pmatrix}$, then $ST = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} = Q$. It follows from (2.14)-(2.17), (2.20)-(2.24), (2.27) and (2.37) that

$$\begin{aligned} &\begin{pmatrix} a_{11}(J) & a_{12}(J) \\ a_{21}(J) & 0 \end{pmatrix} = \delta(Q) = \delta(S)T + S\delta(T) \\ &= \begin{pmatrix} -a_{11}(MW) - MC & (I_{\mathcal{A}} - MW)D + b_{12}(M) \\ C(I_{\mathcal{A}} - MW) & CM \end{pmatrix} \begin{pmatrix} J & 0 \\ WJ & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} I_{\mathcal{A}} - MW & M \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11}(J) - DWJ & JD \\ CJ + c_{21}(W) & WJD \end{pmatrix} \\ &= \begin{pmatrix} \begin{matrix} -a_{11}(MW) + b_{12}(M)WJ + Mc_{21}(WJ) \\ +a_{11}(J) - MWa_{11}(J) \end{matrix} & JD \\ CJ & 0 \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned}0 &= -a_{11}(MW)J + b_{12}(M)WJ + Mc_{21}(WJ) - MWa_{11}(J)\\ &= -a_{11}(MW)J + b_{12}(M)WJ + Mc_{21}(W)J.\end{aligned}$$

Since J is invertible in $\mathcal{A}$, we have that

$$-a_{11}(MW) + b_{12}(M)W + Mc_{21}(W) = 0. \tag{2.47}$$

For any $W, W' \in \mathcal{W}$ and $M \in \mathcal{M}$, by (2.35), (2.46) and (2.47), we have that

$$\begin{aligned}0 &= c_{21}(WMW') - d_{22}(WM)W' - WMc_{21}(W')\\ &= c_{21}(W)MW' + Wa_{11}(MW') - d_{22}(WM)W' - WMc_{21}(W')\\ &= c_{21}(W)MW' + W(b_{12}(M)W' + Mc_{21}(W')) - d_{22}(WM)W' - WMc_{21}(W')\\ &= (c_{21}(W)M + Wb_{12}(M) - d_{22}(WM))W'.\end{aligned} \tag{2.48}$$

For any $W, W' \in \mathcal{W}$ and $M \in \mathcal{M}$, by (2.32), (2.35) and (2.47), we have that

$$\begin{aligned}0 &= b_{12}(M'WM) - b_{12}(M')WM - M'd_{22}(WM)\\ &= a_{11}(M'W)M + M'Wb_{12}(M) - b_{12}(M')WM - M'd_{22}(WM)\\ &= b_{12}(M')WM + M'c_{21}(W)M + M'Wb_{12}(M) - b_{12}(M')WM - M'd_{22}(WM)\\ &= M'(c_{21}(W)M + Wb_{12}(M) - d_{22}(WM)).\end{aligned} \tag{2.49}$$

It follows from (2.10) and (2.25) that a_{11} is a linear mapping derivable at J from $\mathcal{A}$ into $\mathcal{M}_1$ and d_{22} is a linear mapping derivable at zero from $\mathcal{B}$ into $\mathcal{M}_4$.

For any $A \in \mathcal{A},\ M \in \mathcal{M},\ W \in \mathcal{W},\ B \in \mathcal{B}$, it follows from (2.14), (2.15), (2.17), (2.20),(2.22), (2.24), (2.27), (2.28), (2.31), (2.37), (2.38) and (2.41) that

$$\begin{array}{llll} & b_{11}(M) = -MC, & c_{11}(W) = -DW, & d_{11}(B) = 0,\\ a_{12}(A) = AD, & & c_{12}(W) = 0, & d_{12}(B) = -DB,\\ a_{21}(A) = CA, & b_{21}(M) = 0, & & d_{21}(B) = -BC,\\ a_{22}(A) = 0, & b_{22}(M) = CM, & c_{22}(W) = WD. & \end{array}$$

Thus (2.1) holds.

By (2.32), (2.35), (2.42) and (2.46)–(2.49), we have that (2.2)–(2.8) hold. □

Theorem 2.2. *Suppose that* $\mathcal{U} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{W} & \mathcal{B} \end{pmatrix}$ *is a unital block algebra which satisfies that for any A in* $\mathcal{A}$, *there is some integer n such that* $nI_{\mathcal{A}} - A$ *is invertible in* $\mathcal{A}$, $\mathcal{V} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_3 & \mathcal{M}_4 \end{pmatrix}$ *is a unital block* $\mathcal{U}$*-bimodule. Let* $Q = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$, *where* $J \in \mathcal{A}$ *and J is invertible in* $\mathcal{A}$. *If one of the following conditions holds:*

(a) $\mathcal{W}M_1 = 0$ *for* $M_1 \in \mathcal{M}_1$ *implies* $M_1 = 0$ *and* $\mathcal{M}M_4 = 0$ *for* $M_4 \in \mathcal{M}_4$ *implies* $M_4 = 0$;
(b) $M_1\mathcal{W} = 0$ *for* $M_1 \in \mathcal{M}_1$ *implies* $M_1 = 0$ *and* $M_4\mathcal{M} = 0$ *for* $M_4 \in \mathcal{M}_4$ *implies* $M_4 = 0$;
(c) $\mathcal{W}M_1 = 0$ *for* $M_1 \in \mathcal{M}_1$ *implies* $M_1 = 0$ *and* $M_4\mathcal{W} = 0$ *for* $M_4 \in \mathcal{M}_4$ *implies* $M_4 = 0$;
(d) $\mathcal{M}M_4 = 0$ *for* $M_4 \in \mathcal{M}_4$ *implies* $M_4 = 0$ *and* $M_1\mathcal{M} = 0$ *for* $M_1 \in \mathcal{M}_1$ *implies* $M_1 = 0$;

(e) *every linear mapping derivable at J from $\mathcal{A}$ into $\mathcal{M}_1$ is a derivation and $M_4\mathcal{W} = 0$ for $M_4 \in \mathcal{M}_4$ implies $M_4 = 0$;*

(f) *every linear mapping derivable at J from $\mathcal{A}$ into $\mathcal{M}_1$ is a derivation and $\mathcal{M}M_4 = 0$ for $M_4 \in \mathcal{M}_4$ implies $M_4 = 0$,*

then every linear mapping derivable at Q from $\mathcal{U}$ into $\mathcal{V}$ is a derivation.

Proof. Let δ be a linear mapping derivable at Q from $\mathcal{U}$ into $\mathcal{V}$. It follows from Lemma 2.1 that there are elements $D \in \mathcal{M}_2, C \in \mathcal{M}_3$ and linear mappings $a_{11} : \mathcal{A} \to \mathcal{M}_1, b_{12} : \mathcal{M} \to \mathcal{M}_2, c_{21} : \mathcal{W} \to \mathcal{M}_3, d_{22} : \mathcal{B} \to \mathcal{M}_4$ such that a_{11} is a linear mapping derivable at J, d_{22} is a linear mapping derivable at zero and (2.2)–(2.8) hold.

We assume that (a) holds. The proofs for the other cases are analogous.

For any $W \in \mathcal{W}, A_1, A_2 \in \mathcal{A}$, it follows from (2.5) that

$$\begin{aligned} c_{21}(W)A_1A_2 + Wa_{11}(A_1A_2) &= c_{21}(WA_1A_2) = c_{21}(WA_1)A_2 + WA_1a_{11}(A_2) \\ &= c_{21}(W)A_1A_2 + Wa_{11}(A_1)A_2 + WA_1a_{11}(A_2). \end{aligned}$$

Thus

$$W(a_{11}(A_1A_2) - a_{11}(A_1)A_2 - A_1a_{11}(A_2)) = 0$$

for any $W \in \mathcal{W}, A_1, A_2 \in \mathcal{A}$. By (a), we have that

$$a_{11}(A_1A_2) = a_{11}(A_1)A_2 + A_1a_{11}(A_2) \tag{2.50}$$

for any $A_1, A_2 \in \mathcal{A}$.

For any $M \in \mathcal{M}, B_1, B_2 \in \mathcal{B}$, it follows from (2.2) that

$$\begin{aligned} b_{12}(M)B_1B_2 + Md_{22}(B_1B_2) &= b_{12}(MB_1B_2) \\ &= b_{12}(MB_1)B_2 + MB_1d_{22}(B_2) \\ &= b_{12}(M)B_1B_2 + Md_{22}(B_1)B_2 + MB_1d_{22}(B_2). \end{aligned}$$

Thus

$$M(d_{22}(B_1B_2) - d_{22}(B_1)B_2 - B_1d_{22}(B_2)) = 0$$

for any $M \in \mathcal{M}, B_1, B_2 \in \mathcal{B}$. By (a), we have that

$$d_{22}(B_1B_2) = d_{22}(B_1)B_2 + B_1d_{22}(B_2) \tag{2.51}$$

for any $B_1, B_2 \in \mathcal{B}$.

By (a) and (2.8), we have that

$$d_{22}(WM) = c_{21}(W)M + Wb_{12}(M) \tag{2.52}$$

for any $M \in \mathcal{M}, W \in \mathcal{W}$.

By (2.1)–(2.6), (2.50), (2.51) and (2.52), we have that δ is a derivation from $\mathcal{U}$ into $\mathcal{V}$. $\Box$

The triangular algebra is firstly introduced in [5,6]. In [18], Wong firstly discusses the triangular ring. Now we give the definition of triangular algebra.

Let $\mathcal{A}, \mathcal{B}$ be two unital algebras and $\mathcal{M}$ be any unital $(\mathcal{A}, \mathcal{B})$-bimodule. The algebra

$$\mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} : A \in \mathcal{A}, M \in \mathcal{M}, B \in \mathcal{B} \right\}$$

under the usual matrix addition and formal matrix multiplication will be called a *triangular algebra*. It is clear to see that a triangular algebra can be view as a unital block algebra.

From Lemma 2.1 and Theorem 2.2, we have the following statement.

Corollary 2.3. *Suppose that $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a triangular algebra such that for any A in $\mathcal{A}$, there is some integer n such that $nI_{\mathcal{A}} - A$ is invertible in $\mathcal{A}$, $\mathcal{V} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_3 & \mathcal{M}_4 \end{pmatrix}$ is a unital block $\mathcal{U}$-bimodule and $Q = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$, where $J \in \mathcal{A}$ is invertible in $\mathcal{A}$. If one of the following conditions holds:*

(a) $\mathcal{M}M_4 = 0$ for $M_4 \in \mathcal{M}_4$ implies $M_4 = 0$ and $M_1\mathcal{M} = 0$ for $M_1 \in \mathcal{M}_1$ implies $M_1 = 0$;
(b) every linear mapping derivable at J from $\mathcal{A}$ into $\mathcal{M}_1$ is a derivation and $\mathcal{M}M_4 = 0$ for $M_4 \in \mathcal{M}_4$ implies $M_4 = 0$, then every linear mapping derivable δ at Q from $\mathcal{U}$ into $\mathcal{V}$ is a derivation.

Remark 2.4. It is easy to check that if we replace unital algebras by unital rings and replace linear mappings by additive mappings, then Lemma 2.1, Theorem 2.2 and Corollary 2.3 are still true. In this case, Corollary 2.3 generalizes [3, Theorem 3.1].

If $\mathcal{A}$ is a unital (real or complex) Banach algebra with identity I and A is any operator in $\mathcal{A}$, then $nI - A$ is invertible whenever the positive integer $n > \|A\|$. By Theorem 2.2 and Remark 2.4, we have the following corollary which generalizes the main result in [16].

Corollary 2.5. *Let $\mathcal{A}$ be a unital Banach algebra and let $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. Suppose that P is a nontrivial idempotent element in $\mathcal{A}$ and $Q \in P\mathcal{A}P$ is invertible in $P\mathcal{A}P$ which satisfies one of the following conditions:*

(a) For $M \in \mathcal{M}$, $(I-P)\mathcal{A}PMP = 0$ implies $PMP = 0$ and $P\mathcal{A}(I-P)M(I-P) = 0$ implies $(I-P)M(I-P) = 0$;
(b) For $M \in \mathcal{M}$, $PMP\mathcal{A}(I-P) = 0$ implies $PMP = 0$ and $(I-P)M(I-P)\mathcal{A}P = 0$ implies $(I-P)M(I-P) = 0$;
(c) For $M \in \mathcal{M}$, $(I-P)\mathcal{A}PMP = 0$ implies $PMP = 0$ and $(I-P)M(I-P)\mathcal{A}P = 0$ implies $(I-P)M(I-P) = 0$;
(d) For $M \in \mathcal{M}$, $P\mathcal{A}(I-P)M(I-P) = 0$ implies $(I-P)M(I-P) = 0$ and $PMP\mathcal{A}(I-P) = 0$ implies $PMP = 0$;
(e) every linear mapping derivable at Q from $P\mathcal{A}P$ into $P\mathcal{M}P$ is a derivation and for $M \in \mathcal{M}$, $(I-P)M(I-P)\mathcal{A}P = 0$ implies $(I-P)M(I-P) = 0$;
(f) every linear mapping derivable at Q from $P\mathcal{A}P$ into $P\mathcal{M}P$ is a derivation and for $M \in \mathcal{M}$, $P\mathcal{A}(I-P)M(I-P) = 0$ implies $(I-P)M(I-P) = 0$.

If δ is a linear mapping derivable at Q from $\mathcal{A}$ into $\mathcal{M}$, then δ is a derivation.

3. Application 1. Self-adjoint operator algebras

In this section, we consider that H is a complex Hilbert space with $\dim H \geqslant 2$ and denote $B(H)$ the set of all bounded linear operators on H.

Lemma 3.1. *Let $\mathcal{A}$ be a unital prime Banach algebra with a nontrivial idempotent P and let $Q \in P\mathcal{A}P$ be invertible in $P\mathcal{A}P$. If δ is a linear mapping derivable at Q from $\mathcal{A}$ into itself, then δ is a derivation.*

A *von Neumann algebra* is a self-adjoint unital subalgebra of $B(H)$ closed in the weak operator topology. By [10, Theorem 5.5.2], every von Neumann algebra $\mathcal{A}$ is the closed linear span of its projections. Thus if $\mathcal{A} \neq \mathbb{C}I$, then there exist nontrivial projections in $\mathcal{A}$. Since every factor von Neumann algebra is prime, by Lemma 3.1, we have the following corollary.

Corollary 3.2. *Let $\mathcal{A} \neq \mathbb{C}I$ be a factor von Neumann algebra on H and P be any nontrivial idempotent in $\mathcal{A}$. Then for any $Q \in P\mathcal{A}P$ which is invertible in $P\mathcal{A}P$, every linear mapping derivable at Q from $\mathcal{A}$ into itself is a derivation. In particular, if $\mathcal{A} = B(H)$, then every linear mapping derivable at Q from $B(H)$ into itself is a derivation.*

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. We denote the set of $n \times n$ matrices whose entries are elements in $\mathcal{A}$ by $\mathrm{Mat}_n(\mathcal{A})$.

Corollary 3.3. *Let $n \in \mathbb{N}$ and $n \geqslant 2$. If $\mathcal{A}$ is a unital Banach algebra and $\mathcal{M}$ is a unital $\mathrm{Mat}_n(\mathcal{A})$-bimodule, then there exists a nontrivial idempotent P in $\mathrm{Mat}_n(\mathcal{A})$ such that for any $Q \in P\mathrm{Mat}_n(\mathcal{A})P$ which is invertible in $P\mathrm{Mat}_n(\mathcal{A})P$, every linear mapping derivable at Q from $\mathrm{Mat}_n(\mathcal{A})$ into $\mathcal{M}$ is a derivation.*

Proof. Let $P = \begin{pmatrix} I_{\mathcal{A}} & 0 \dots 0 \\ 0 & 0 \dots 0 \\ \dots & \\ 0 & 0 \dots 0 \end{pmatrix}$. Then we can write $\mathcal{U} = \mathrm{Mat}_n(\mathcal{A})$ as

$$\begin{pmatrix} P\mathcal{U}P & P\mathcal{U}(I_{\mathcal{U}} - P) \\ (I_{\mathcal{U}} - P)\mathcal{U}P & (I_{\mathcal{U}} - P)\mathcal{U}(I_{\mathcal{U}} - P) \end{pmatrix}$$

and $\mathcal{M}$ as

$$\begin{pmatrix} P\mathcal{M}P & P\mathcal{M}(I_{\mathcal{U}} - P) \\ (I_{\mathcal{U}} - P)\mathcal{M}P & (I_{\mathcal{U}} - P)\mathcal{M}(I_{\mathcal{U}} - P) \end{pmatrix}.$$

Thus we have that $P\mathcal{U}(I_{\mathcal{U}} - P) = \begin{pmatrix} 0 & \mathcal{A} \dots \mathcal{A} \\ 0 & 0 \dots 0 \\ \dots & \\ 0 & 0 \dots 0 \end{pmatrix}$ and $(I_{\mathcal{U}} - P)\mathcal{U}P = \begin{pmatrix} 0 & 0 \dots 0 \\ \mathcal{A} & 0 \dots 0 \\ \dots & \\ \mathcal{A} & 0 \dots 0 \end{pmatrix}$. Let $T_m \in \mathcal{U}$ satisfy that the element in entry $(1, m)$ is $I_{\mathcal{A}}$ and other entries are zero. Thus $T_m \in P\mathcal{A}(I_{\mathcal{U}} - P)$ when $m \geqslant 2$. Let $T_m{}^* \in \mathcal{U}$ satisfy that the entry at $(m, 1)$ is $I_{\mathcal{A}}$ and other entries are zero. Thus $T_m{}^* \in P^{\perp}\mathcal{U}P$ when $m \geqslant 2$. Let $F_m = T_m{}^*T_m \in \mathcal{U}$. Then F_m satisfies that the element in entry (m, m) is $I_{\mathcal{A}}$ and other entries are zero. Thus $\sum_{m=2}^{n} F_m = I_{\mathcal{U}} - P$.

For any $M \in \mathcal{M}$ with $P\mathcal{U}(I_{\mathcal{U}} - P)M(I_{\mathcal{A}} - P) = 0$, since $T_m \in P\mathcal{U}(I_{\mathcal{U}} - P)$ when $m \geqslant 2$, we have $T_mM(I_{\mathcal{U}} - P) = 0$. Thus $(I_{\mathcal{U}} - P)M(I_{\mathcal{U}} - P) = \sum_{m=2}^{n} F_mM(I_{\mathcal{U}} - P) = \sum_{m=2}^{n}(T_m)^*T_mM(I_{\mathcal{U}} - P) = 0$.

For any $M \in \mathcal{M}$ with $PMP\mathcal{A}(I_{\mathcal{U}} - P) = 0$, since $T_m \in P\mathcal{U}(I_{\mathcal{U}} - P)$ when $m \geqslant 2$, we have $PMT_m = 0$. It is easy to see that $T_mT_m{}^* = P$. Thus $PMP = PMT_mT_m{}^* = 0$.

Of course, $P\mathcal{U}P$ is still a Banach algebra with identity P. Then for any $Q \in P\mathrm{Mat}_n(\mathcal{A})P$ which is invertible in $P\mathrm{Mat}_n(\mathcal{A})P$, it follows from Theorem 2.2 that every linear mapping derivable at Q from $\mathrm{Mat}_n(\mathcal{A})$ into $\mathcal{M}$ is a derivation. □

Theorem 3.4. *Let $\mathcal{A}$ be a non-abelian von Neumann algebra on H. Then there exists a nontrivial projection P in $\mathcal{A}$ such that for any $Q \in P\mathcal{A}P$ which is invertible in $P\mathcal{A}P$, every linear mapping derivable at Q from $\mathcal{A}$ into any unital Banach $\mathcal{A}$-bimodule $\mathcal{M}$ is a derivation.*

Proof. By type decomposition, see [11, Chapter 6], every von Neumann algebra can be written as the direct sum of Type I_n, Type II_1, Type II_∞ and Type III von Neumann algebras. Since $\mathcal{A}$ is a non-abelian von Neumann algebra, we have that there exists a central projection P_0 in $\mathcal{A}$ such that $P_0\mathcal{A}P_0$ is a von Neumann algebra on P_0H with no abelian summands.

Case 1. If $P_0\mathcal{A}P_0$ is not of Type I_2, then by [17, Lemma 14], there exist nontrivial orthogonal projections P_1 and P_2 in $P_0\mathcal{A}P_0$ with $P_1 + P_2 = P_0$ and there is a projection $\widetilde{P}_2 \leqslant P_2$ such that $P_1 \sim \widetilde{P}_2$ via $V \in \mathcal{A}$. That is $V^*V = P_1$ and $VV^* = \widetilde{P}_2$.

Let $P = I_{\mathcal{A}} - P_1$.

For $M \in \mathcal{M}$ with $(I_{\mathcal{A}} - P_1)M(I_{\mathcal{A}} - P_1)\mathcal{A}P = 0$, we have

$$(I_{\mathcal{A}} - P_1)M(I_{\mathcal{A}} - P_1) = P_1MP_1 = P_1MP_1V^*VP_1 = P_1MP_1V^*\widetilde{P}_2(I_{\mathcal{A}} - P_1))VP_1 = 0.$$

By [10, Theorem 5.5.7], $P\mathcal{A}P$ is still a von Neumann algebra on PH. By [13, Corollary 2.5], for any $Q \in P\mathcal{A}P$ which is invertible in $P\mathcal{A}P$, every linear mapping derivable at Q from $P\mathcal{A}P$ into $P\mathcal{M}P$ is a Jordan derivation. It follows from [1, Theorem 6.10] that every Jordan derivation from $P\mathcal{A}P$ into $P\mathcal{M}P$ is a derivation. Thus every linear mapping derivable at Q from $P\mathcal{A}P$ into $P\mathcal{M}P$ is a derivation.

By Corollary 2.5, we have that every linear mapping derivable at P from $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule M is a derivation.

Case 2. If $P_0\mathcal{A}P_0$ is of Type I_2, then by the definition of Type I_2, there exist nontrivial orthogonal projections P_1 and P_2 in $P_0\mathcal{A}P_0$ with $P_1 + P_2 = P_0$ and $P_1 \sim P_2$. The same as the proof of case 1, we complete the proof. $\square$

Remark 3.5. By Theorem 3.4, we solve 'almost all of' von Neumann algebras except the case of abelian von Neumann algebras. The following example shows that for any abelian von Neumann algebra, there is even no nontrivial idempotent P in $\mathcal{A}$ such that every linear mapping derivable at P from $\mathcal{A}$ into itself is a derivation.

Example 3.6. Let $\mathcal{A}$ be an abelian von Neumann algebra on H. Let P be any nontrivial idempotent in $\mathcal{A}$. Let δ be a linear mapping from $\mathcal{A}$ into itself defined by

$$\delta(A) = A(I_{\mathcal{A}} - P)$$

for any $A \in \mathcal{A}$.

Let $A, B \in \mathcal{A}$ with $AB = P$. Since $\mathcal{A}$ is abelian, we have that $\delta(AB) = 0 = \delta(A)B + A\delta(B)$. Hence δ is a linear mapping derivable at P from $\mathcal{A}$ into itself. But it is easy to see that δ is not a derivation.

4. Application 3. Non-self-adjoint operator algebras

In this section, we consider that H is a complex Hilbert space with $\dim H \geqslant 2$, unless stated otherwise.

By a *subspace lattice* on H, we mean a collection $\mathcal{L}$ of closed subspaces of H with $(0), H \in \mathcal{L}$ and such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\cap M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\vee M_r$ denotes the closed linear span of $\{M_r\}$. For any subspace lattice $\mathcal{L}$, we define $\mathrm{alg}\mathcal{L}$ by

$$\mathrm{alg}\mathcal{L} = \{T \in B(H), TN \subseteq N, \forall N \in \mathcal{L}\}.$$

It is not difficult to show that $\mathrm{alg}\mathcal{L}$ is closed in operator-norm, and is a unital Banach algebra. For convenience, we disregard the distinction between a closed subspace and the orthogonal projection onto it.

A subspace lattice $\mathcal{L}$ on a Hilbert space H is called a *commutative subspace lattice*, or a *CSL*, if the projections in $\mathcal{L}$ commute with each other. If $\mathcal{L}$ is a commutative subspace lattice, then $\mathrm{alg}\mathcal{L}$ is called a *CSL* algebra. Recall that a CSL algebra $\mathrm{alg}\mathcal{L}$ is *irreducible* if and only if $(\mathrm{alg}\mathcal{L})' = \mathbb{C}I$.

A subspace lattice $\mathcal{L}$ is said to be *completely distributive* if for every family $\{X_{i,j}\}_{i\in I, j\in J}$ of elements in $\mathcal{L}$,

$$\bigwedge_{i\in I}\bigvee_{j\in J} x_{i,j} = \bigvee_{f\in J_I}\bigwedge_{i\in I} x_{i,f(i)} \quad \text{and} \quad \bigvee_{i\in I}\bigwedge_{j\in J} x_{i,j} = \bigwedge_{f\in J_I}\bigvee_{i\in I} x_{i,f(i)},$$

where J_I denotes the set of all maps from I into J. If $\mathcal{L}$ is a completely distributive commutative subspace lattice (CDCSL), then alg$\mathcal{L}$ is called a *CDCSL algebra*.

Definition 4.1. Let $\mathcal{A}$ be an associative algebra and suppose that £ is a linear sub-manifold of $\mathcal{A}$. We say that £ is a *triple nilpotent commutator Lie ideal* if

(a) £ is triple nilpotent, i.e., $[£, [\mathcal{A}, £]] = \{0\}$;
(b) every element in £ is a commutator, i.e.,$£ \subseteq \{[A, B] : A, B \in \mathcal{A}\}$;
(c) £ is a Lie ideal, i.e., $[\mathcal{A}, £] \subseteq £$.

If alg$\mathcal{L}$ is a CSL algebra and P is a projection in $\mathcal{L}$, then $P(\text{alg}\mathcal{L})P^{\perp}$ is an obvious example of triple nilpotent commutator Lie ideal.

The following two lemmas are obtained in [15].

Lemma 4.2 [15, Corollary 3.3]**.** *Let* alg$\mathcal{L}$ *be a CDCSL algebra on H. Then there exists a projection P in* $\mathcal{L}$ *such that* $P(\text{alg}\mathcal{L})P^{\perp}$ *is a maximal triple nilpotent commutator Lie ideal.*

Lemma 4.3 [15, Theorem 3.4]**.** *Let* alg$\mathcal{L}$ *be an irreducible CDCSL algebra on H and suppose that P is a nontrivial projection in* $\mathcal{L}$*. Then the following are equivalent.*

(a) P is faithful. (That is, for $T \in \text{alg}\mathcal{L}$, $TP(\text{alg}\mathcal{L})P^{\perp} = \{0\}$ *implies* $TP = 0$ *and* $P(\text{alg}\mathcal{L})P^{\perp}T = \{0\}$ *implies* $P^{\perp}T = 0$.*)*
(b) $P(\text{alg}\mathcal{L})P^{\perp}$ *is a maximal triple nilpotent commutator Lie ideal.*
(c) For every $Q \in \mathcal{L}$ *with* $Q_- \not\geq P$*, there exists a nonzero vector* $y_Q \in H$ *such that* $x \otimes y_Q \in P(\text{alg}\mathcal{L})P^{\perp}$ *for all* $x \in Q$*. Dually, for every* $Q \in \mathcal{L}$ *with* $Q \not\leq P$*, there exists a nonzero vector* $x_Q \in H$ *such that* $x_Q \otimes y \in P(\text{alg}\mathcal{L})P^{\perp}$ *for all* $y \in Q_-^{\perp}$.

Here $Q_- = \vee\{L \in \mathcal{L} : Q \not\leq L\}$.

Lemma 4.4. *Let* alg$\mathcal{L}$ *be an irreducible CDCSL algebra on H and suppose that P is a nontrivial projection in* $\mathcal{L}$*. Then the following are equivalent.*

(a) P is faithful.
(b) For $T \in B(H)$, $TP(\text{alg}\mathcal{L})P^{\perp} = \{0\}$ *implies* $TP = 0$ *and* $P(\text{alg}\mathcal{L})P^{\perp}T = \{0\}$ *implies* $P^{\perp}T = 0$.
(c) For $T \in B(H)$, $PTP(\text{alg}\mathcal{L})P^{\perp} = \{0\}$ *implies* $PTP = 0$ *and* $P(\text{alg}\mathcal{L})P^{\perp}TP^{\perp} = \{0\}$ *implies* $P^{\perp}TP^{\perp} = 0$.

Proof. (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are obviously. We only need to prove that (a) $\Rightarrow$ (b).

Let $T \in B(H)$ with $TP(\text{alg}\mathcal{L})P^{\perp} = \{0\}$. By Lemma 4.3, for every $Q \in \mathcal{L}$ with $Q_- \not\geq P$, there exists a nonzero vector $y_Q \in H$ such that $x \otimes y_Q \in P(\text{alg}\mathcal{L})P^{\perp}$ for all $x \in Q$. Thus $Tx \otimes y_Q = 0$. For each $t \in H$, $(y_Q, t)Tx = 0$. Let $t = y_Q$. Thus $Tx = 0$, for any $x \in Q$. Since $\mathcal{L}$ is completely distributive, by [14, Theorem 5.1 and Lemma 5.1], $P = \vee\{Q \in \mathcal{L} : Q_- \not\geq P\}$. Thus $TP = 0$.

Let $T \in B(H)$ with $P(\text{alg}\mathcal{L})P^{\perp}T = \{0\}$. By Lemma 4.3, for every $Q \in \mathcal{L}$ with $Q \not\leq P$, there exists a nonzero vector $x_Q \in H$ such that $x_Q \otimes y \in P(\text{alg}\mathcal{L})P^{\perp}$ for all $y \in Q_-^{\perp}$. Thus $x_Q \otimes yT = 0$. For each $t \in H$, $(y, Tt)x_Q = 0$. Since $x_Q \neq 0$, we have $T^*y = 0$ for any $y \in Q_-^{\perp}$. Since $\mathcal{L}$ is completely distributive, by [14, Theorem 5.2], $P = \wedge\{Q_- : Q \in \mathcal{L}, Q \not\leq P\}$. We have that $P^{\perp} = \vee\{Q_-^{\perp} : Q \in \mathcal{L}, Q \not\leq P\}$. Thus $T^*P^{\perp} = 0$. Hence $P^{\perp}T = 0$. $\square$

Theorem 4.5. *Let* $\mathcal{L}$ *be an irreducible CDCSL on H. Then there exists a nontrivial projection P in* alg$\mathcal{L}$ *such that for any* $Q \in P\text{alg}\mathcal{L}P$ *which is invertible in* $P\text{alg}\mathcal{L}P$*, every linear mapping derivable at Q from* alg$\mathcal{L}$ *into* $B(H)$ *is a derivation.*

Proof. Let δ be a linear mapping derivable at P from alg$\mathcal{L}$ into $B(H)$.
Case 1. If $\mathcal{L}$ is trivial, i.e., $\mathcal{L} = \{0, H\}$. Then alg$\mathcal{L} = B(H)$. Since dim $H \geqslant 2$, there is a nontrivial projection P in B(H). By Corollary 3.2, P satisfies the statement.
Case 2. If $\mathcal{L}$ is nontrivial. By Lemma 4.2, there is a projection P in $\mathcal{L}$ such that $P(\text{alg}\mathcal{L})P^{\perp}$ is a maximal triple nilpotent commutator Lie ideal. Since $\mathcal{L}$ is nontrivial, we have P is nontrivial. By Lemmas 4.3 and 4.4, for any $T \in B(H)$, $PTP(\text{alg}\mathcal{L})P^{\perp} = \{0\}$ implies $PTP = 0$ and $P(\text{alg}\mathcal{L})P^{\perp}TP^{\perp} = \{0\}$ implies $P^{\perp}TP^{\perp} = 0$. Thus it follows from Corollary 2.5 that δ is a derivation. $\square$

For any subspace lattice $\mathcal{L}$ on H, we denote $\mathcal{L}^{\perp} = \{L^{\perp}, L \in \mathcal{L}\}$.

Theorem 4.6. *Let $\mathcal{L}$ be a CDCSL on H. Suppose that there exists a projection P in $\mathcal{L} \cap \mathcal{L}^{\perp}$ such that $P\mathcal{L}$ is nontrivial and Palg$\mathcal{L}P$ is a irreducible CDCSL algebra. Then there is a nontrivial projection P_0 in* alg$\mathcal{L}$ *such that for any $Q \in P_0\text{alg}\mathcal{L}P_0$ which is invertible in $P_0\text{alg}\mathcal{L}P_0$, every linear mapping derivable at Q from* alg$\mathcal{L}$ *into itself is a derivation.*

Proof. Since $P\mathcal{L}$ is nontrivial and $P\text{alg}\mathcal{L}P$ is an irreducible CDCSL algebra. By Lemmas 4.2 and 4.3, there is a nontrivial projection P_1 in $P\mathcal{L}$ such that for any $T \in \text{alg}\mathcal{L}$, $P_1PTPP_1P(\text{alg}\mathcal{L})P(P - P_1) = \{0\}$ implies $P_1PTPP_1 = 0$. Since $P \in \mathcal{L} \cap \mathcal{L}^{\perp}$, we have that for any $T \in \text{alg}\mathcal{L}$, $P_1TP_1(\text{alg}\mathcal{L})(I - P_1) = \{0\}$ implies $P_1TP_1 = 0$.

Since $\mathcal{L}$ is a CSL on H and $P \in \mathcal{L} \cap \mathcal{L}^{\perp}$, we have that $P_1 \in P\mathcal{L} \subseteq \mathcal{L}$. Thus $(I - P_1)(\text{alg}\mathcal{L})(I - P_1)$ is also a CSL algebra on $(I - P_1)H$. By [13, Corollary 2.6], for any $Q \in (I - P_1)\text{alg}\mathcal{L}(I - P_1)$ which is invertible in $(I - P_1)\text{alg}\mathcal{L}(I - P_1)$, every linear mapping derivable at Q from $(I - P_1)(\text{alg}\mathcal{L})(I - P_1)$ into itself is a derivation.

Let $P_0 = I - P_1$. Thus it follows from Corollary 2.5 that every linear mapping derivable at P_0 from alg$\mathcal{L}$ into itself is a derivation. $\square$

Remark 4.7. By the proof of [7, Theorem 4.1], we know that if $\mathcal{L}$ is a CDCSL on a separable complex Hilbert space H, then there are no more than countably many pairwise orthogonal projections $\{P_n : n \in \Lambda\}$ in $\mathcal{L} \cap \mathcal{L}^{\perp}$ such that $\text{alg}\mathcal{L} = \sum_n \oplus(\text{alg}\mathcal{L})P_n$, where each $(\text{alg}\mathcal{L})P_n$ is viewed as a subalgebra of operators acting on P_nH is an irreducible CDCSL algebra. By Theorem 4.6, we get the result of 'almost all of' CDCSLs on a separable complex Hilbert space H except the case that all of $\mathcal{L}P_n$ are trivial. In the following, we will discuss it.

By Theorem 3.4, we have the following corollary.

Corollary 4.8. *Suppose that $\{P_n : n \in \Lambda\}$ are no more than countably many pairwise orthogonal projections in $B(H)$ satisfying $\sum_n P_n = I$ and there is some $n_0 \in \Lambda$ such that* $\text{dim}P_{n_0} \geqslant 2$. *Let $\mathcal{A} = \sum_n \oplus B(P_nH)$. Then there exists a nontrivial projection P in $\mathcal{A}$ such that for any $Q \in P\mathcal{A}P$ which is invertible in $P\mathcal{A}P$, every linear mapping derivable at Q from $\mathcal{A}$ into itself is a derivation.*

Example 3.6 tells us that in the case that both all of $\mathcal{L}P_n$ are trivial and all of $\text{dim}P_nH$ are 1, the result is not true. Thus we finish the question about CDCSL algebras.

At the end, we answer the question for another kind of CSL algebras. We give the definitions firstly. Let $\mathcal{L}$ be a CSL. If $P, Q \in \mathcal{L}$, then $Q - P$ is called an *interval*. Nests included in $\mathcal{L}$ are *independent* if the product of nonzero intervals, which are taken from each nest, is again nonzero. In [12], Laurie gives an example of a CSL generated by two independent nests which is not completely distributive. Thus the following theorem is independent to Theorem 4.6 and it generalizes the main result in [20].

Theorem 4.9. *Let $\mathcal{L}$ be a nontrivial CSL generated by finitely many independent nests $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on a separable complex Hilbert space H and let $P \in \mathcal{L} \cup \mathcal{L}^{\perp}$ be any nontrivial projections, i.e.,$P \neq 0$ and I. Then for any $Q \in P\text{alg}\mathcal{L}P$ which is invertible in $P\text{alg}\mathcal{L}P$, every linear mapping derivable at Q from* alg$\mathcal{L}$ *into $B(H)$ is a derivation.*

Proof

Case 1. $P \in \mathcal{L}$.

By [19, Lemma 2.4], for any $A \in B(H)$, $PAP\mathrm{alg}\mathcal{L}P^{\perp} = 0$ implies $PAP = 0$ and $P\mathrm{alg}\mathcal{L}P^{\perp}AP^{\perp} = 0$ implies $P^{\perp}AP^{\perp} = 0$.

Since $\mathrm{alg}\mathcal{L}$ is a unital Banach algebra, it follows from Corollary 2.5 that for any $Q \in P\mathrm{alg}\mathcal{L}P$ which is invertible in $P\mathrm{alg}\mathcal{L}P$, every linear mapping derivable at Q from $\mathrm{alg}\mathcal{L}$ into $B(H)$ is a derivation.

Case 2. $P \in \mathcal{L}^{\perp}$, i.e., $P^{\perp} \in \mathcal{L}$.

By [19, Lemma 2.4], for any $A \in B(H)$, $P^{\perp}AP^{\perp}\mathrm{alg}\mathcal{L}P = 0$ implies $P^{\perp}AP^{\perp} = 0$ and $P^{\perp}\mathrm{alg}\mathcal{L}PAP = 0$ implies $PAP = 0$.

Since $\mathrm{alg}\mathcal{L}$ is a unital Banach algebra, it follows from Corollary 2.5 that for any $Q \in P\mathrm{alg}\mathcal{L}P$ which is invertible in $P\mathrm{alg}\mathcal{L}P$, every linear mapping derivable at Q from $\mathrm{alg}\mathcal{L}$ into $B(H)$ is a derivation. $\square$

It is easy to see that [20, Theorem 3.1] is just a corollary of Theorem 4.9 and we do not need to assume the linear mapping derivable at Q is continuous.

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